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THE DIFFRACTION OF PLANE ELASTIC WAVES BY A DELAMINATED RIGID INCLUSION WHEN THERE IS SMOOTH CONTACT IN THE DELAMINATION REGION†

V. G. POPOV

Odessa

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A solution of the problem of the diffraction of harmonic elastic waves by a thin rigid strip-like delaminated inclusion in an unbounded elastic medium, in which the conditions for plane deformation are satisfied, is proposed. We mean by a delaminated inclusion an inclusion, one side of which is completely bonded to the elastic medium, while the second does not interact in any way with it, or this interaction is partial. It is assumed that the conditions for smooth contact are satisfied in the delamination region. The method of solution is based on the use of previously constructed discontinuous solutions of the equations describing the vibrations of an elastic medium under plane deformation conditions. The problem therefore reduces to solving a system of three singular integral equations in the unknown stress and strain jumps at the inclusion. An approximate solution of the latter enabled formulae to be obtained that are convenient for numerical realization when investigating the stressed state in the region of the inclusion and its displacements when acted upon by incident waves.

1. Suppose an elastic medium, which is in a state of plane deformation, contains a thin rigid inclusion, situated in the xy plane in the section $y = 0, |x| \leqslant a$. At $y = -0$ it is completely bonded to the elastic medium, and at $y = +0$ it is peeled off, and at this boundary the conditions for smooth contact are satisfied. This means that the following boundary conditions are satisfied on the inclusion

$$\tau_{yx}(x,+0)=0,\ v(x,\pm 0)=\delta_1+\gamma x,\ u(x,-0)=\delta_2\ |x|\leqslant a \tag{1.1}$$

Moreover, the stresses and strains undergo discontinuities along the line on which the inclusion is situated. We will denote the jumps at the discontinuities as follows:

$$\langle\sigma_y\rangle=\chi_1(x),\ \langle\tau_{yx}\rangle=\chi_2(x),\ \langle v\rangle=0,\ \langle u\rangle=\chi_4(x)\,|x|\leqslant a \tag{1.2}$$

The angular brackets in (1.2) mean the same as in [1]. Plane longitudinal or transverse waves interact with the inclusion. The waves are specified by the following potentials

$$\varphi_0(x,y)=\frac{A}{\varkappa_1}e_1(x,y),\ \psi_0(x,y)=\frac{B}{\varkappa_2}e_2(x,y) \tag{1.3}$$

$$e_k(x,y)=\exp[i\varkappa_k(x\cos\theta_0+y\sin\theta_0),\ \varkappa_1^2=\frac{\rho\omega^2}{\lambda+2\mu},\ \varkappa_2^2=\frac{\rho\omega^2}{\mu}$$

where θ_0 is the angle between the direction of the wave propagation and the x axis, and ω is the oscillation frequency. The factor $e^{-i\omega t}$ is omitted everywhere.

The constants δ_1, δ_2, γ describe the translational and rotational motions of the inclusion and remain to be determined. To do this the following equalities are obtained from the equations of motion of the inclusion as a rigid body

$$m\delta_j\omega^2=-\int\chi_j(x)dx,\ j=1,2;\ \frac{4}{3}ma^2\omega^2\gamma=\int x\chi_1(x)dx \tag{1.4}$$

where m is the mass per unit length of the inclusion, and integration over x (and below over η also) is carried out in the section $[-a, a]$.

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In addition we have the equality

$$\int \chi_4'(x)dx = 0 \tag{1.5}$$

To solve the problem we will represent the resultant displacement and stress field in the following form

$$T(x, y) = T_1(x, y) + T_0(x, y) \tag{1.6}$$
$$T_k(x,y) = [t_j^k(x,y)] = [\sigma_y^k, \tau_{yx}^k, \upsilon^k, u^k], \quad k = 0,1$$

The vector $T_1(x, y)$ is defined by the discontinuous solution (1.2) from [1] with jumps (1.2), while the components of the vector $T_0(x,y)$ are the stresses and displacements due to the longitudinal or transverse wave (1.3) propagating in the medium and are given by (2.11) and (2.12) from [1].

The displacement and stress field in the medium is obviously uniquely defined by (1.6), after the unknown jumps (1.2) have been determined. To do this, as in [1], we replace conditions (1.1) by equivalent conditions, which, after substituting (1.6) into them, take the form

$$t_2^1(x,+0) = -t_2^0(x,+0), \quad t_3^{1\prime}(x,\pm 0) = \gamma - t_3^{0\prime}(x,\pm 0)$$
$$t_4^{1\prime}(x,-0) = -t_4^{0\prime}(x,-0), \quad |x| \leqslant a \tag{1.7}$$
$$t_3^1(-a,\pm 0) = \delta_1 - \gamma a - t_3^0(-a,\pm 0), \quad t_4^1(-a,-0) = \delta_2 - t_4^0(-a,-0)$$

We substitute the values of the components of the vector T_1, defined in (1.2) [1] into the first three equations of (1.7). Here we must take into account the following formulae for the limiting values, which arise from the properties of discontinuous solutions

$$T_1^*(x,\pm 0) = \pm\frac{1}{2}Z(\eta) + \frac{1}{2\pi}\int\frac{AZ(\eta)}{\eta - x}d\eta + \frac{1}{2\pi}\int W(\eta - x)Z(\eta)d\eta \tag{1.8}$$

where, in the case considered, $Z(\eta) = [\chi_1(\eta), \chi_2(\eta), 0, \chi'_4(\eta)]^T$ and A is a numerical matrix, the elements of which are expressed in terms of the elasticity constants of the medium, containing the inclusion, while the elements of the functional matrix $W(z) = \{W_{jk}\}$ $(j, k = 1, 2, 3, 4)$ are found from the formulae

$$W_{jj}(z) = W_{j5-j}(z) = 0 \tag{1.9}$$
$$W_{jk}(z) = \sum_m A_{jk}^m\left(\frac{z}{2}\right)^{2m-1} + \ln|z|\sum_m B_{jk}^m\left(\frac{z}{2}\right)^{2m-1}, \quad j,k = 1,2,3,4$$

Here and below the summation over m is carried out from 1 to ∞. When obtaining (1.9) we used formulae (1.3)–(1.5) from [1], and also the well-known representations for Hankel functions in the form of power series [2].

As a result of the relatively unknown jumps, a system of three singular integral equations is obtained, the matrix form of which is (integration over τ is carried out in the interval $[-1, 1]$)

$$B\Phi + Q\Gamma\Phi + R\Phi = F \tag{1.10}$$
$$\Gamma\Phi = \frac{1}{2\pi}\int\frac{\Phi(\tau)}{\tau - t}d\tau, \quad R\Phi = \frac{1}{2\pi}\int R(\tau - t)\Phi(\tau)d\tau$$
$$\Phi(\tau) = [\varphi_1(\tau), \varphi_2(\tau), \varphi_3(\tau)]^T, \quad F(t) = [f_1(t), f_2(t), f_3(t)]^T$$
$$\varphi_1(\tau) = \mu^{-1}\chi_1(a\tau), \quad \varphi_2(\tau) = \mu^{-1}\chi_2(a\tau), \quad \varphi_3(\tau) = \chi_4'(a\tau)$$
$$f_1(t) = -\mu^{-1}\tau_{yx}^0(at,+0), \quad f_2(t) = \gamma - \upsilon_x^{0\prime}(at,0), \quad f_3(t) = -u_x^{0\prime}(at,0)$$

The non-zero elements of the third-order matrices B, Q and R are

$$b_{12}=\tfrac{1}{2},\ b_{31}=-\tfrac{1}{2},\ q_{11}=q_{23}=-\xi^2,\ q_{13}=2(1-\xi^2)$$
$$q_{21}=q_{23}=-(1+\xi^2)/2,\ \xi=c_1/c_2$$
$$R_{11}(\tau-t)=aW_{21}[a(\tau-t)],\ R_{13}(\tau-t)=a\mu W_{24}[a(\tau-t)]$$
$$R_{21}(\tau-t)=aW_{13}[a(\tau-t)],\ R_{23}(\tau-t)=aW_{24}[a(\tau-t)]$$
$$R_{32}(\tau-t)=aW_{42}[a(\tau-t)]$$

We similarly obtain from the remaining equalities (1.7)

$$\frac{1}{2\pi}\int\varphi_1(\tau)[-q_{32}\ln(\tau+1)+\Pi_{11}(\tau+1)]d\tau+$$
$$+\frac{1}{2\pi}\int\varphi_2(\tau)[-q_{23}\ln(\tau+1)+\Pi_{12}(\tau+1)]d\tau=\delta_{01}-\gamma+f_{01} \tag{1.11}$$
$$\frac{1}{2\pi}\int\varphi_2(\tau)[-q_{13}\ln(\tau+1)+\Pi_{22}(\tau+1)]d\tau=\delta_{02}+f_{02}$$
$$\Pi_{jk}(\tau+1)=\sum_m p_{jk}^m\left(\frac{\tau+1}{2}\right)^{2m}+\ln\frac{\tau+1}{2}\sum_m h_{jk}^m\left(\frac{\tau+1}{2}\right)^{2m}$$

2. To solve system (1.10) simultaneously with (1.11) we multiply both sides of (1.10) by the matrix Q^{-1} (the latter exists since $\det Q\neq 0$). We obtain

$$C\Phi+E\Gamma\Phi+Q^{-1}R\Phi=Q^{-1}F \tag{2.1}$$

where $C+Q^{-1}B$ and E is the third-order identity matrix. After this, as was done in [3] when solving a similar problem in a static formulation, we introduce the unknown functions $\psi_j(\tau)$ ($j=1,2,3$), defined by the equalities

$$\Psi(\tau)=P^{-1}\Phi(\tau),\ \ \Phi(\tau)=P\Psi(\tau) \tag{2.2}$$
$$\Psi(\tau)=[\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)]^T$$

The matrix P is constructed in such a way that $P^{-1}CP=D$, while D is a third-order diagonal matrix. Since the eigenvalues of the matrix C are different, this matrix exists and can easily be constructed [4]. Here the diagonal elements of the matrix D are

$$d_{11}=\lambda_1=0,\ d_{22}=\lambda_2=-\tfrac{1}{2},\ d_{33}=\lambda_3=\tfrac{1}{2}$$

Substituting (2.2) into (2.1) and multiplying the equation obtained by P^{-1}, we arrive at the following system of singular integral equations (summation over i is carried out from 1 to 3)

$$\lambda_k\psi_k(t)+\frac{1}{2\pi}\int\frac{\psi_k(\tau)}{\tau-t}d\tau+\frac{1}{2\pi}\sum_i\int L_{ki}(\tau-t)\psi_i(\tau)d\tau=c_k\gamma+g_k(t),\ t\in[-1,1],\ k=1,2,3 \tag{2.3}$$
$$L=\{L_{kj}\}=P^{-1}Q^{-1}RP,\ \ G=\{g_k\}=PQ^{-1}F$$
$$c_1=-4q_{21}^{-1},\ c_2=2q_{11}q_{21}^{-1},\ c_3=-2q_{11}$$

It is necessary to consider system (2.3) simultaneously with Eqs (1.11), (1.4) and (1.5), which, after introducing the new unknowns (2.2), contain integrals with a logarithmic singularity

$$I_k=\int\psi_k(\tau)\ln(\tau+1)d\tau,\ \ k=1,2,3 \tag{2.4}$$

The solution of system (2.3) in the class of functions having integrable singularities [5, 6], can be expanded in the form

$$\psi_k(\tau)=\omega_k(\tau)w_k(\tau),\ \ \omega_k(\tau)=(1-\tau)^{\alpha_k}(1+\tau)^{\beta_k}\ \ k=1,2,3 \tag{2.5}$$

where α_k is the root of the equation

$$\lambda_k + \operatorname{ctg}\alpha_k/2 = 0,\ -1 < \alpha_k < 0,\ \alpha_k + \beta_k = -1$$

In the case of system (2.3) we obtain

$$\alpha_1 = \beta_1 = -\tfrac{1}{2},\ \alpha_2 = \beta_3 = -\tfrac{1}{4},\ \alpha_3 = \beta_2 = -\tfrac{3}{4}$$

We approximate the functions $w_k(t)$ by polynomials of the best approximation

$$w_k(t) = G_{nk}(t) = \sum_m w_{km} \frac{P_n^{\alpha_k,\beta_k}(t)}{(t-\tau_{km})[P_n^{\alpha_k,\beta_k}(\tau_{km})]'},\quad k = 1,2,3 \tag{2.6}$$

$w_{km} = w_k(\tau_{km})$, $P_n^{\alpha_k,\beta_k}(t)$ are Jacobi polynomials of the nth degree, orthogonal with weight $\omega_k(t)$ and τ_{km} are the roots of these polynomials.

If $\omega_k(t)$ are represented by formulae (2.5) and (2.6), we have the following quadrature formulae for the singular integral operators [7]

$$\lambda_k \psi_k(t_{kj}) + \frac{1}{2\pi}\int \frac{\psi_k(\tau)}{\tau - t_{kj}}d\tau = \sum_m b_{km}\frac{w_{km}}{\tau_{km} - t_{kj}} \tag{2.7}$$

$$b_{km} = -\frac{\pi P_n^{-\alpha_k,-\beta_k}(\tau_{km})}{2\sin\pi\alpha_k[P_n^{\alpha_k,\beta_k}(\tau_{km})]'},\quad k = 1,2,3$$

Approximate values can be obtained for integrals (2.4) if we replace $\omega_k(t)$ using formulae (2.5) and (2.6). As a result we obtain

$$I_k = \sum_m \sigma_{km}^n w_{km},\quad \sigma_{km}^n = \int \frac{\ln(\tau+1)P_n^{\alpha_k,\beta_k}(\tau)\omega_k(\tau)}{(\tau-\tau_{km})[P_n^{\alpha_k,\beta_k}(\tau_{km})]'}d\tau,\quad k = 1,2,3 \tag{2.8}$$

The integrals σ_{km}^n can be evaluated by a method specially developed for singular integrals, which contain orthogonal polynomials, described in [3].

If we now use relations (2.7) and (2.8) and the regular integrals are replaced by sums using the Gauss–Jacobi quadrature formulae with appropriate weight [8] $\omega_k(\tau)$ $(k = 1, 2, 3)$, we obtain the following system of linear algebraic equations

$$\begin{aligned}
&\sum_m b_{km}\frac{w_{km}}{\tau_{km} - t_{kj}} + \frac{1}{2\pi}\sum_i\sum_m b_{im}w_{im}L_{ki}(\tau_{im} - t_{kj}) = \\
&= c_k\gamma + g_k(t_{kj}),\quad j = 1,2,\dots,n-1;\ k = 1,2,3 \\
&\frac{\sigma_{11}}{8\pi}\sum_m \sigma_{1m}^n w_{1m} + \frac{1}{2\pi}\sum_i\sum_m b_{im}w_{im}C_{1i}(1+\tau_{im}) = \delta_{01} - \gamma + f_{01} - \\
&-\frac{\sigma_{11}}{8\pi}\sum_m \sigma_{2m}^n w_{2m} - \frac{1}{8\pi}\sum_m \sigma_{3m}^n w_{3m} + \\
&+\frac{1}{2\pi}\sum_i\sum_m b_{im}w_{im}C_{2i}(1+\tau_{im}) = \delta_{02} + f_{02} \\
&-\sum_m b_{1m}w_{1m} + \beta_{21}\sum_m b_{2m}w_{2m} - \frac{\beta_{21}}{\sigma_{11}}\sum_m b_{3m}w_{3m} = 4\delta_{01}m_0k_0^2 \\
&-\sum_m \tau_{1m}b_{1m}w_{1m} + \beta_{21}\sum_m \tau_{2m}b_{2m}w_{2m} - \frac{\beta_{21}}{\sigma_{11}}\sum_m \tau_{3m}b_{3m}w_{3m} = \frac{16}{3}\gamma m_0\varkappa_0^2 \\
&\sum_m b_{2m}w_{2m} + \sigma_{11}^{-1}\sum_m b_{3m}w_{3m} = 4\delta_{02}m_0\varkappa_0^2 - \sigma_{22}\sum_m b_{2m}w_{2m} + \sum_m b_{3m}w_{3m} = 0
\end{aligned} \tag{2.9}$$

Solving system (2.9) we obtain w_{km} ($k = 1, 2, 3$; $m = 1, 2, \ldots, n$) and δ_{01}, δ_{02}, γ, which enables us to investigate the displacements and stresses in the medium containing the inclusion numerically.

We will take as the quantity characterizing the stress concentration in the region of the ends of the inclusion, as in [9, 10], the coefficients of the singularities of the stress jumps at the inclusion, which can be calculated from the formulae

$$K_j^{\pm} = \lim_{\tau\to\pm1}(1\pm\tau)^{3/4}\frac{\chi_j(a\tau)}{\mu} = \lim_{\tau\to\pm1}(1\pm\tau)^{3/4}\varphi_j(\tau);\quad j=1,2$$

After substituting (2.5) and (2.6) into this equation and taking the limit we obtain

$$K_1^{\pm} = 2^{-5/4}\beta_{21}S^{\pm},\quad K_2^{\pm} = \pm 2^{-5/4}S^{\pm};\quad S^{+} = G_{n2}(1),\quad S^{-} = G_{n3}(-1) \tag{2.10}$$

It can be seen that the stressed state in the elastic medium near the ends of the inclusion is determined by $S^{\pm}$.

The results of a numerical analysis of the displacements of the inclusion are shown in Figs 1 and 2. In Fig. 1 we show the absolute values of the dimensionless amplitudes of the vibrations of the inclusions $|\delta_{01}|$, $|\delta_{02}|$, $|\gamma|$ as a function of the dimensionless frequency $\varkappa_0 = \varkappa_2 a$. Curve 1 shows the change in $|\delta_{01}|$ when a longitudinal wave interacts with the inclusion. It is assumed that the wave propagates at an angle $\theta_0 = \pi/2$ (the wave is incident on the bonded side of the inclusion) and at an angle $\theta_0 = 3\pi/2$ (the wave is incident on the peeling side of the inclusion). In both cases the values of $|\delta_{01}|$ are the same, while $|\delta_{02}| = |\gamma| = 0$. Curves 2 show the change in $|\delta_{02}|$ when $\theta_0 = \pi/2$ (the continuous curve) and $\theta_0 = 3\pi/2$ (the dashed curve). The change in $|\gamma|$ when a transverse wave is incident on the inclusion at an angle $\theta_0 = \pi/2$ and $\theta_0 = 3\pi/2$ is represented by curves 3, where $|\delta_{01}| = 0$.

The results of an investigation of the amplitude of the vibrations of the inclusion as a function of the angle of incidence of the wave for a fixed frequency $\varkappa_0 = 2$ are shown in Fig. 2, where the continuous curves correspond to a longitudinal wave incident on the inclusion while the dashed curves correspond to the incidence of a transverse wave. Curves 1–3 show the change in $|\delta_{01}|$, $|\delta_{02}|$, $|\gamma|$. Where longitudinal waves propagate $|\delta_{01}$ has its maximum values when $|\delta_{02}|$, while $\theta_0 = \pm\pi/4$ and $|\gamma|$ have their maximum values when $\theta_0 = \pm\pi/4$. If transverse waves propagate, $|\delta_{01}|$ reaches a maximum when $\theta_0 = \pm\pi/4$, $|\delta_{02}|$ reaches a maximum $\theta_0 = \pm\pi/2$ and $|\gamma|$ reaches a maximum when $\theta_0 = 0$.

Figure 3 shows the absolute values of the coefficients of the stress singularities $|S^{\pm}|$ as a function of frequency. Curves 1 and 2 represent this relationship when a transverse wave, propagating at angles of $\theta_0 = \pi/2$ and $\theta_0 = 3\pi/2$, respectively, is diffracted by the inclusion. In this case $S^{+} = S^{-}$. The presence of a maximum when $\varkappa_0 > 2$ is of interest. When longitudinal waves, incident at the same angles, are diffracted, the values of S^{+} and S^{-} are close to zero. Curves 3 illustrate the change in $|S^{+}|$ as the frequency increases for transverse waves, while curves 4 illustrate the change for longitudinal waves incident at

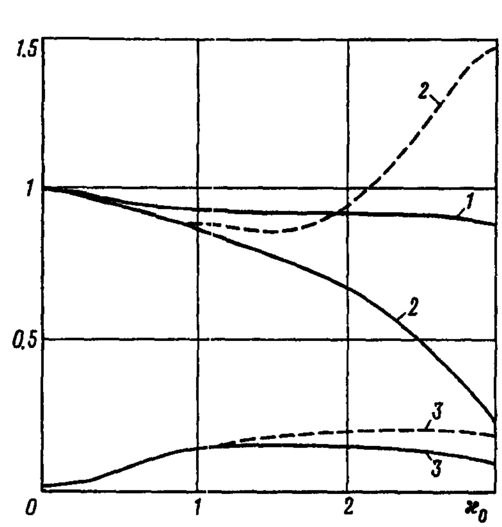

Fig. 1.

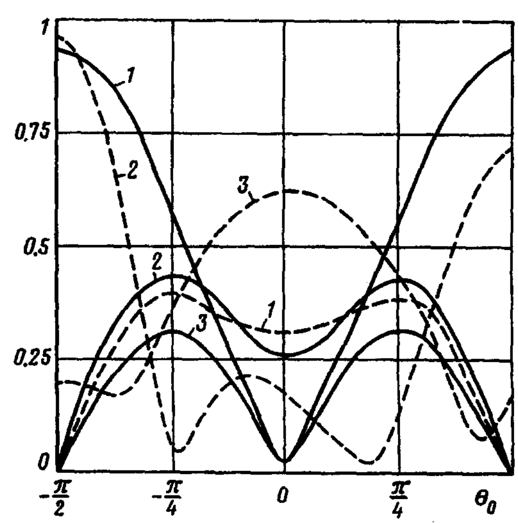

Fig. 2.

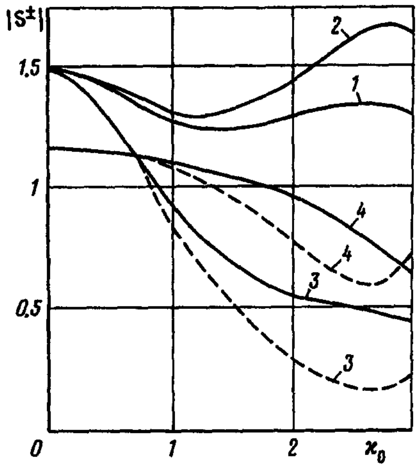

Fig. 3.

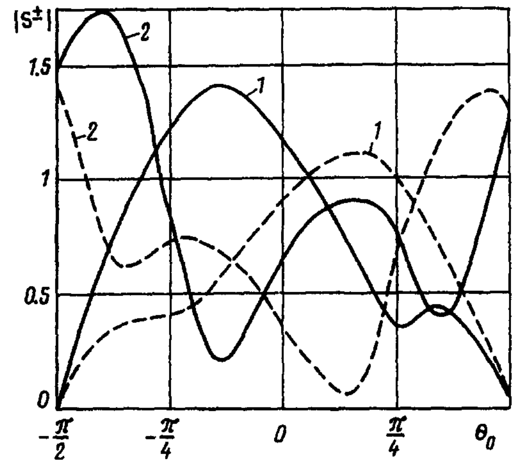

Fig. 4.

an angle $\theta_0 = 0$. The continuous curves show the change in $| S^- |$ and the dashed curves represent the change in $| S^+ |$.

In Fig. 4 we show graphs of $| S^- |$ (the continuous curves) and $| S^+ |$ (the dashed curves) against the angle of incidence θ_0 for longitudinal waves (curves 1) and transverse waves (curves 2) for a fixed frequency $\varkappa_0 = 2$. When longitudinal waves are diffracted the maximum values of $| S^{\pm} |$ occur at $\theta_0 = \pm 3\pi/20$, while the minimum values occur at $\theta_0 = \pm \pi/2$. If a transverse wave interacts with the inclusion, the greatest values of $| S^{\pm} |$ are observed when $\theta_0 = \pm 2\pi/5$, and the least values are observed when $\theta_0 = \pm 3\pi/20$.

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